Advanced Quantum Field Theory (QFT II), Summer 2016 | **Exercise Sheet 1**

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Exercise 1: Grassmann numbers

We consider a set of variables $G = \{\theta_1, ..., \theta_n\}$ satisfying anticommutator-relations

$$\{\theta_i, \theta_j\} = 0 \text{ for } i, j \in \{1, ..., n\}.$$

They are called Grassmann numbers (or Grassmann variables), in contrast to commuting numbers in $\mathbb{R}$, $\mathbb{C}$ etc. which are sometimes called c-numbers (c for 'commuting').

a) Show that $\theta_i^2 = 0$ for $\theta_i \in G$.

b) Show that $\{\eta_1, \eta_2\} = 0$ with $\eta_1 = a_1\theta_i + b_1\theta_j$, $\eta_2 = a_2\theta_k + b_2\theta_l$, where $a_1, a_2, b_1, b_2 \in \mathbb{C}$ and all the θ are in G.

c) For a power series in a Grassmann variable θ, the integration over θ can be defined by

$$\int d\theta = 0 \text{ and } \int d\theta\, \theta = 1.$$

For multiple integrals, an ambiguous sign is fixed by the convention

$$\int d\theta_2 \left(\int d\theta_1\, \theta_1 \right) \theta_2 = +1.$$

Show that $\int d\theta\, \mathrm{e}^{ib\theta} = ib$ for $b \in \mathbb{C}$.

d) With Grassmann numbers θ_1, θ_2 we define complex Grassmann numbers by

$$\theta = \frac{1}{\sqrt{2}} (\theta_1 + i\theta_2)\,,\ \theta^\star = \frac{1}{\sqrt{2}} (\theta_1 - i\theta_2)\,.$$

Show that $\int d\theta^\star \int d\theta\, \mathrm{e}^{-\theta^\star b \theta} = b$ and $\int d\theta^\star \int d\theta\, \theta\theta^\star \mathrm{e}^{-\theta^\star b\theta} = 1$.

e) Under a variable transformation $\theta_i \to \theta_i' = \sum_{j=1}^n A_{ij}\theta_j$, where A_{ij} are c-number entries in an $n \times n$ matrix A, the integral measure of a multiple integral transforms as

$$d\theta_1 d\theta_2 ... d\theta_n = \det(A)\, d\theta_1' d\theta_2' ... d\theta_n'.$$

Compute the integral $\int d\theta_1^\star \int d\theta_1 \int d\theta_2^\star \int d\theta_2 ... \int d\theta_n^\star \int d\theta_n \mathrm{e}^{-\sum_{i=1}^n \sum_{j=1}^n \theta_i^\star M_{ij} \theta_j}$, where M_{ij} are the c-number entries of an antisymmetric $n \times n$ matrix M.

Exercise 2: Wick's theorem and functional differentiation

Consider the theory of the free Klein-Gordon field $\phi(x)$ with the Feynman propagator $D_F(x-y)$. Let $J(x)$ be some other scalar field (a so-called source term).
a) Use Wick's theorem to compute $\langle 0|T(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4))|0\rangle$ in terms of Feynman propagators. Draw the corresponding Feynman graphs.
b) Use the definition of functional differentiation to show that

$$\frac{\delta}{\delta J(x)} \mathrm{e}^{i \int d^4y J(y)\phi(y)} = i\phi(x)\mathrm{e}^{i \int d^4y J(y)\phi(y)}.$$

c) Consider the functional

$$Z[J] = \mathrm{e}^{-\frac{1}{2}\int d^4x \int d^4y J(x) D_F(x-y) J(y)}$$

and let $Z_0 = Z[0]$. Compute

$$\left((-i)^2 \frac{1}{Z_0} \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} Z[J] \right)\Bigg|_{J=0}$$

and

$$\left((-i)^4 \frac{1}{Z_0} \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \frac{\delta}{\delta J(x_3)} \frac{\delta}{\delta J(x_4)} Z[J] \right)\Bigg|_{J=0}$$

in terms of Feynman propagators. Draw the corresponding Feynman graphs and give a brief interpretation of these results.